Physics 216,Quiz 2

December 17, 2012, 90 minutes

1. Find the Fourier Series of the function $f(x)=\begin{cases} 0, & -\pi<x<0 \\ x, & 0<x<\pi \end{cases}$

2. The Fourier series of the function of period 2 which is equal to x on the interval $(-1,1)$ is given by
$$f(x)=-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\sin n\pi x$$
Use this expansion and Parseval theorem to show that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

3. Given the two recursion relations for Legendre polynomials
$$xP_l'(x)-P_{l-1}'(x)=lP_l(x)$$
$$P_l'(x)-xP_{l-1}'(x)=lP_{l-1}(x)$$
show that
$$\left(1-x^2\right)P_l'(x)=lP_{l-1}(x)-lxP_l(x)$$
Differentiate this last equation with respect to x and eliminate $P_{l-1}'(x)$ to show that
$$\left(1-x^2\right)P_l''(x)-2xP_l'(x)+l(l+1)P_l(x)=0$$

4. Expand the function $f(x)=P_7'(x)$ in terms of Legendre polynomials. Hint: Use the following recursion relation repeatedly $(2l+1)P_l(x)=P_{l+1}'(x)-P_{l-1}'(x)$ and $P_1(x)=x$.

5. Use the recursion relation for Hermite polynomials
$$2xH_n(x)=H_{n+1}(x)+2nH_{n-1}(x)$$
and orthogonality condition $\int\limits_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)\,dx=\delta_{nm}\sqrt{\pi}2^n n!$ to evaluate the integral
$$\int\limits_{-\infty}^{\infty}x^2e^{-x^2}H_n(x)H_n(x)\,dx=\sqrt{\pi}2^n n!\left(n+\frac{1}{2}\right)$$

6. (a) Using the definition $P_l^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x)$ Show that
$$P_l^{m+1}(x)=\sqrt{1-x^2}\frac{d}{dx}P_l^m(x)+\frac{1}{\sqrt{1-x^2}}mxP_l^m(x)$$
(b) Define $L_+=e^{i\varphi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\right)$ to show that $L_+Y_l^m$ is proportional to Y_l^{m+1} using $Y_l^m\sim e^{im\varphi}P_l^m(\cos\theta)$ and the change of variable $x=\cos\theta$.

7. Use the relation $xL_n^k = (2n+k+1)\,L_n^k - (n+k)\,L_{n-1}^k - (n+1)\,L_{n+1}^k$ to show that

$$\int_0^\infty e^{-x}x^{k+1}L_n^k(x)\,L_n^k(x)\,dx = \frac{(n+k)!}{n!}\,(2n+k+1)$$

using the orthogonality condition $\int_0^\infty e^{-x}x^k L_n^k(x)\,L_m^k(x)\,dx = \frac{(n+k)!}{n!}\delta_{nm}$..

8. Find the Fourier transform $F(\omega)$ of the function $f(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$ .Use the convolution theorem $\int_{-\infty}^{\infty} f(t)\,f^*(t)\,dt = \int_{-\infty}^{\infty} F(\omega)\,F^*(\omega)\,d\omega$ to evaluate the integral $\int_0^\infty \frac{\sin^2\omega}{\omega^2}d\omega$.